Chapter 3

Day 2: Matrix Transformations

3.1 *Schedule*

- 0900-0915: Debrief
- 0915-0945: Synthesis
- 0945-1030: 2D Rotations
- 1030-1045: Coffee
- 1045-1130: 3D Rotations
- 1130-1200: Reflections and Shearing
- 1200-1220: Review and Preview
- 1220-1230: Survey

3.2 *Debrief*

- With your table-mates, identify a list of key concepts/take home messages/things you learned in the assignment. Try to group them in categories like "Concepts", "Technical Details", "Matlab", etc.
- Try to resolve your confusions with your table-mates and by talking to an instructor.

3.3 *Synthesis*

Exercise 3.1

These are fundamental ideas about matrices and it is important to complete these. They should be done by hand.

1. What is the difference between a scalar, a vector, a matrix, and an array?
2. What are the rules for adding matrices?
3. When can two matrices be multiplied, and what is the size of the output?
4. What is the distributive property for matrix multiplication?
5. What is the associative property for matrix multiplication?
6. What is the commutative property for matrix multiplication?

Exercise 3.2

These are synthesis problems. It would be helpful to complete these. They should be done by hand.

1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that $\mathbf{A}^2$ commutes with $\mathbf{A}$.
2. Use the distribution law to expand $(\mathbf{A} + \mathbf{B})^2$ assuming that $\mathbf{A}$ and $\mathbf{B}$ are matrices of appropriate size. How does this compare to the situation for real numbers?
3. Show that $\mathbf{D} = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$ satisfies the matrix equation $\mathbf{D}^2 - \mathbf{D} - 6\mathbf{I} = \mathbf{0}$.

Exercise 3.3

These are challenge problems. Pick one of them to wrestle with. It is not important to complete these. They should be done by hand.

1. The matrix exponential is defined by the power series

$$\exp \mathbf{A} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

   Assume $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Find a formula for $\exp \mathbf{A}$.
2. The real number 0 has just one square root: 0. Show, however, that the 2×2 zero matrix has infinitely many square roots by finding all 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{0}$.
3. Use induction to prove that $\mathbf{A}^n$ commutes with $\mathbf{A}$ for any square matrix $\mathbf{A}$ and positive integer n.

3.4 2D Rotation Matrices

We're going to think about how to use rotation matrices to rotate a geometrical object. In doing so we will solidify fundamental concepts around matrix multiplication and start to explore the notion of "inverse". For clarity we will first work in 2D. Recall that the rotation matrix $\mathbf{R}(\theta)$:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

will rotate an object counterclockwise **about the origin** through an angle of θ.

Exercise 3.4

This is a hands-on, conceptual problem involving the multiplication of 2D rotation matrices.

1. Place an object on your table, and imagine that the origin of an xy-coordinate system is at the center of your object with $+z$ pointing upwards.
2. Rotate it counterclockwise by 30 degrees, and then again by another 60 degrees. What is it's orientation now? How would you get there in one rotation instead? What does this suggest about the multiplication of rotation matrices?
3. What happens if you first rotate it by 60 degrees, and then by 30 degrees? What does this suggest about the commutative property of 2D rotation matrices?

Exercise 3.5

This is an algebra problem involving the multiplication of 2D rotation matrices.

1. Use some algebra to show that 2D rotation matrices commute, i.e. $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_2)\mathbf{R}(\theta_1)$.
2. Use some algebra to show that $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$. You will need to look up some trig identities.

Exercise 3.6

Now, consider a rectangle of width 2 and height 4, centered at the origin. For clarity, this means that the corners of the rectangle have coordinates $(1, 2)$, $(-1, 2)$, $(-1, -2)$, and $(1, -2)$.

1. Plot these four points by hand and connect them with lines to complete the rectangle.
2. Now, using the appropriate rotation matrix, transform each of the corner points by a rotation through 30 degrees counterclockwise (recall that the sin and cos of 30 degrees can be expressed

exactly). Compute and plot the resulting points by hand and connect them with lines. Does the resulting figure look like you'd expect?

Exercise 3.7

Now, let's do it in MATLAB.

1. Create and plot the original 4 points:$(1, 2)$, $(-1, 2)$, $(-1, -2)$, and $(1, -2)$. Then create the matrix that rotates them by 30 degrees counterclockwise, transform each of the four original points using the rotation matrix, and plot the resulting points. Does this look right? *Reminder:* `plot(1,2,'x')` *puts a mark at the point (1,2). Matlab: the functions cos and sin expect radians, while cosd and sind expect degrees.*

2. Operating on individual points with the rotation matrix is cool, but we can be much more efficient by operating on all 4 points at the same time. Write down the matrix whose columns represent the four corners of the rectangle. Then write down the matrix multiplication problem we can solve to transform the rectangle from above all at once. Create these matrices in MATLAB to perform the rotation in a single operation. Plot the resulting matrix to confirm your transformation! *Some MATLAB tips:* `plot(X,Y)` *creates a line plot of the values in the vector Y versus those in the vector X. So if you wanted to plot a line from the origin (0,0) to the point (1,2), you would do this: plot([0 1],[0 2]).* The command `axis([-xlim xlim -ylim ylim])` sets the axes of the current plot to run from `-xlim` to `xlim` and from `-ylim` to `ylim`

3. What is the area of the rectangle before and after the rotation?

4. What matrix should you use to undo this rotation? Define it in MATLAB and check.

5. Show on the board that the product of this matrix with the original rotation matrix is the identity matrix. For clarity, let's give this matrix the symbol $\mathbf{R}^{-1}$. It is the matrix that inverts the original operation and is known as the *inverse* of the matrix $\mathbf{R}$.

3.5 3D Rotations

We can extend the idea of 2D rotations to 3D rotations. The simplest approach is to think of 3D rotations as a composition of rotations about different axes. First let's define the rotation matrices for counterclockwise rotations of angle θ about the x, y and z axes respectively.

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{3.1}$$

$$\mathbf{R}_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3.2}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

For example, to first rotate a vector $\mathbf{v}$ counterclockwise by θ about the x axis followed by counterclockwise by ϕ about the z axis, you need to do the following

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \mathbf{v} \tag{3.4}$$

We will next look at some sequence of physical rotations and relate them to these rotation matrices.

Exercise 3.8

Hold a closed book in front of you, with the top of the book towards the ceiling ($+z = (0, 0, 1)$ direction) and the cover of the book pointed towards you ($+x = (1, 0, 0)$ direction), which leaves the opening side of the book pointing towards your right ($+y = (0, 1, 0)$) and the spine toward the left.

1. Rotate the book by 90 degrees counter-clockwise about the x-axis, then from this position, rotate the book by 90 degrees counter-clockwise about the z-axis. Which direction is the cover of the book facing now?
2. Return to the starting position. Now rotate the book by 90 degrees counter-clockwise about the z axis, and then from this position, rotate the book by 90 degrees counter-clockwise about the x axis. Which direction is the cover of the book facing now? Is it the same as in part a?
3. An operation "commutes" if changing the order of operation doesn't change the result. Do 3D rotations commute?
4. The cover of the book is originally pointed towards $(1, 0, 0)$. Multiply this vector with the appropriate sequence of rotation matrices from above to reproduce your motions from part 1. Do you end up with the correct final cover direction?
5. Multiply the $(1, 0, 0)$ vector with the appropriate sequence of rotation matrices to reproduce the motions from part 2. Do you end up with the correct final cover direction?

6. Multiply the result of the previous part by the appropriate sequence of rotation matrices to return to the original $(1, 0, 0)$ vector.
7. From either of your answers to part 4 or part 5, try, instead of operating on the $(1, 0, 0)$ vector sequentially with one rotation matrix and then the other, take the product of the two rotation matrices first, and then multiply $(1, 0, 0)$ with the resultant matrix. Does this reproduce your answer?
8. Based on your answers to the previous parts, show that $(\mathbf{R}_z\mathbf{R}_x)^{-1} = \mathbf{R}_x^{-1}\mathbf{R}_z^{-1}$. This is a general property of matrix inverses – it works for all square, invertible matrices, not just rotation matrices!

3.6 *Reflection and Shearing*

In this activity we will meet reflection and shearing matrices, which will allow us to explore transformation matrices in general.

Reflection

Exercise 3.9

What do the following *reflection* matrices do? Think about it first, draw some sketches and then test your hypothesis in MATLAB using the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$. How much does the area of your basic rectangle change, if at all? What is the inverse of each?

1. $$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
2. $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
3. $$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Shearing

Exercise 3.10

What do the following *shearing* matrices do? Think about it first, draw some sketches and then test your hypothesis in MATLAB with the rectangle with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$. How much does the area of your basic rectangle change, if at all? What is the inverse of each?

1. $$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & 0 \\ 2k & 1 \end{bmatrix}$$

Review and Preview

Solution 3.1

1. Scalars, vectors, and matrices are examples of arrays. A 0-dimensional array can be thought of as a scalar. A 1-dimensional array is a vector. A 2-dimensional array is a matrix.

2. The matrices have to be the same size and addition is element-wise.

3. The matrices have to compatible (inner dimensions agree), and the output is dictated by the outer dimensions, i.e. $(n \times m)(r \times s) = (n \times s)$.

4. Distributive property: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

5. Associative property: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

6. Commutative property: Two matrices commute if $\mathbf{AB} = \mathbf{BA}$ but this is not always true.

Solution 3.2

1. You need to show that $\mathbf{A}^2\mathbf{A} = \mathbf{AA}^2$ for this particular matrix. You can do it by multiplying.

2. Using the distributive property you can see that $(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$

3. If you plug $\mathbf{D}$ and $\mathbf{D}^2$ into the equation you should find that the result is a zero matrix.

Solution 3.3

1. The matrix exponential is defined by the power series $\exp \mathbf{A} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \ldots$. Notice that this $\mathbf{A}$ is diagonal and $\mathbf{A}^2 = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}$ and the exponential becomes $\exp \mathbf{A} = \begin{bmatrix} 1 + 2 + 2^2/2! + \ldots & 0 \\ 0 & 1 + 3 + 3^2/2! + \ldots \end{bmatrix}$. If you have seen power series before then you will recognise that $\exp \mathbf{A} = \begin{bmatrix} \exp 2 & 0 \\ 0 & \exp 3 \end{bmatrix}$.

2. You can define a general two by two matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find $\mathbf{A}^2$, set each of the entries equal to zero and find constraints on the entries a, b, c, d.

3. You need to show that $\mathbf{A}^n\mathbf{A} = \mathbf{AA}^n$ for any square matrix $\mathbf{A}$ and any positive integer n by induction. First you show it is true for $n = 1$ and $n = 2$. Then assume it is true for some $n = k$, and prove that it must be true for $n = k + 1$. You use the fact that $\mathbf{A}$ commutes with itself and the associative property, i.e $\mathbf{A}^2\mathbf{A} = (\mathbf{AA})\mathbf{A} = \mathbf{A}(\mathbf{AA}) = \mathbf{AA}^2$.

Solution 3.4

1. Okay, I placed my book on the table.

2. You could get there by rotating once by 90 degrees. This suggests that the product of two rotation matrices of angles θ_1 and θ_2 is a rotation matrix of $\theta_1 + \theta_2$, i.e. $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$.

3. You end up in the same orientation so it doesn't matter the order. This suggests that the order of multiplication doesn't matter so that two rotation matrices must commute.

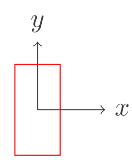
Solution 3.5

1. You could multiply out two rotation matrices with angle θ_1 and θ_2 in the two different orders and you will observe that the output is the same because real numbers commute, i.e. $\cos\theta_1 \cos\theta_2 = \cos\theta_2 \cos\theta_1$.

2. If you multiply two matrices together you will get the following expression in the first row and first column, $\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$. You will find a trig identity which reduces this to $\cos(\theta_1 + \theta_2)$. Similar reductions take place for the other elements.

Solution 3.6

1. The rectangle is



2. The rotation matrix is

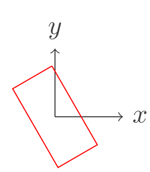
$$\mathbf{R} = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Applying this to each point, we get

$$\mathbf{R}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-2}{2} \\ \frac{1+2\sqrt{3}}{2} \end{bmatrix}, \ \mathbf{R}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+2}{2} \\ \frac{1-2\sqrt{3}}{2} \end{bmatrix},$$

$$\mathbf{R}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{3}-2}{2} \\ \frac{-1+\sqrt{3}}{2} \end{bmatrix}, \ \mathbf{R}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{3}+2}{2} \\ \frac{-1-\sqrt{3}}{2} \end{bmatrix}.$$

And the rotated figure looks like,



Solution 3.7

1. There are lots of ways to do this point by point. Here is an example of how to transform the bottom right point:

```
>> BR = [1;-2]
>> plot(BR(1,:),BR(2,:),'b*')
>> rotmatrix = [cosd(30) -sind(30); sind(30) cosd(30)]
>> nBR = rotmatrix*BR
>> plot(nBR(1,:),nBR(2,:),'r*')
```

2. There are lots of ways to do this. Here is an example where we include the first point twice so that the points can easily be connected with lines:

```
>> pts = [1 -1 -1 1 1;2 2 -2 -2 2]
>> npts = rotmatrix*pts
>> plot(pts(1,:),pts(2,:),'b'), hold on
>> plot(pts(1,:),pts(2,:),'r')
>> axis([-3 3 -3 3])
>> axis equal
```

3. The area of the rectangle is the same before and after rotation: 8 square units.

4. To undo this rotation you could simply rotate it by 30 degrees clockwise, using the matrix

$$\mathbf{R}^{-1} = \begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix}.$$

5. The product of $\mathbf{R}^{-1}$ and $\mathbf{R}$ is

$$\mathbf{R}^{-1}\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where we have used the trig identity $\cos^2\theta + \sin^2\theta = 1$.

Solution 3.8

1. The cover is now facing toward the $+y$ axis (the positive part of the y axis).

2. The cover is now facing the $+z$ axis. This is different than in part a.

3. Since the answers for the first two parts are different, 3D rotations do not commute.

4. Let $\mathbf{v}$ be the vector that represents the initial direction of the cover of the book,

$$\mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Rotation by 90 degrees counterclockwise around the x axis is given by

$$\mathbf{R}_x = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}.$$

so that the new vector becomes

$$\mathbf{R}_x\mathbf{v} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Rotation by 90 degrees counterclockwise around the z axis is given by

$$\mathbf{R}_z = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

so that the new vector becomes

$$\mathbf{R}_z\mathbf{R}_x\mathbf{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

which is the correct final direction.

5. Using the matrices from above,

$$\mathbf{R}_x\mathbf{R}_z\mathbf{v} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

6. To rotate 90 degrees clockwise around the x axis we use the matrix

$$\mathbf{R}_x^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & -1 & 0\end{bmatrix}$$

and to rotate 90 degrees clockwise around the z axis we use the matrix

$$\mathbf{R}_z^{-1} = \begin{bmatrix}0 & 1 & 0\\-1 & 0 & 0\\0 & 0 & 1\end{bmatrix}.$$

Then we can return the vector $(0, 0, 1)$ to its original position $(1, 0, 0)$ by

$$\mathbf{R}_z^{-1}\mathbf{R}_x^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

7. We can multiply the rotation matrices together and perform a single matrix multiplication. For part d, the relevant matrix product is

$$\mathbf{R}_z\mathbf{R}_x = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and we see that

$$\mathbf{R}_z\mathbf{R}_x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as expected.

8. We can see from the previous parts that

$$(\mathbf{R}_z\mathbf{R}_x)^{-1} = \mathbf{R}_x^{-1}\mathbf{R}_z^{-1}.$$

In other words, when you take the inverse, the order of operations must swap!

Solution 3.9

1. This matrix reflects everything over the y-axis. In the figure below, the original blue rectangle becomes the orange rectangle. The area of the rectangle stays the same.

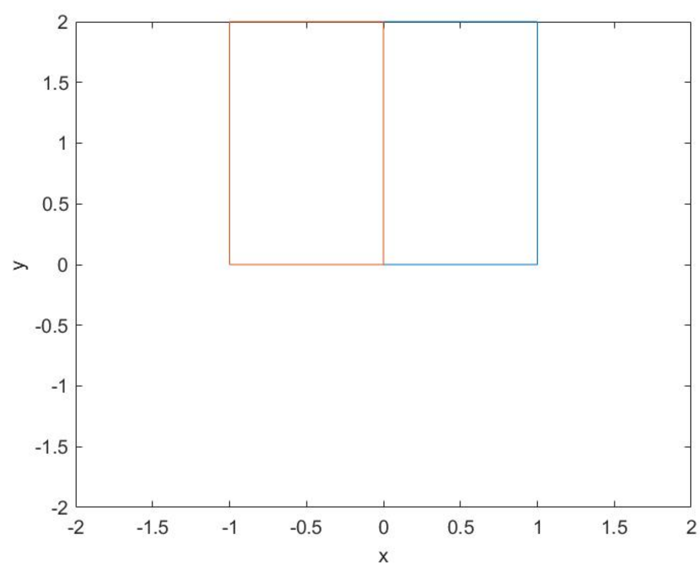


Figure 3.1: Reflection over y-axis.

2. This matrix reflects everything over the x-axis. In the figure below, the original blue rectangle becomes the orange rectangle. The area of the rectangle stays the same.

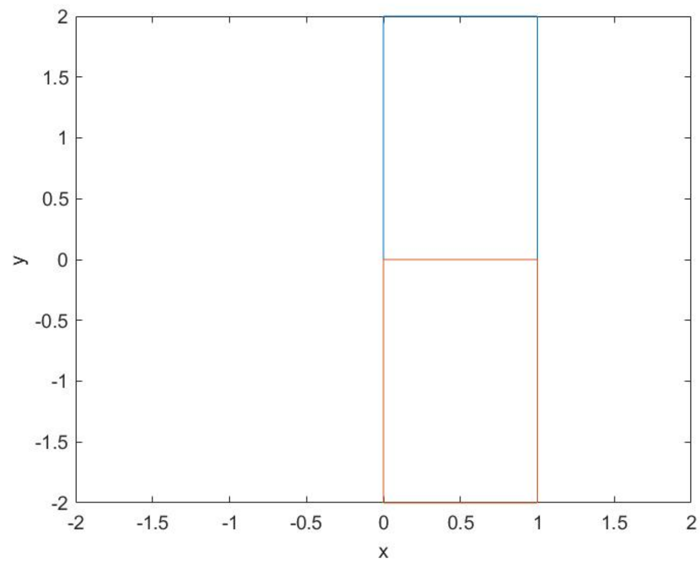


Figure 3.2: Reflection over x-axis.

3. For example, let $\theta = 30$ degrees. Then the rectangle is reflected along the line that is 30 degrees counterclockwise from the x-axis. In the figure below, the original blue rectangle becomes the orange rectangle.

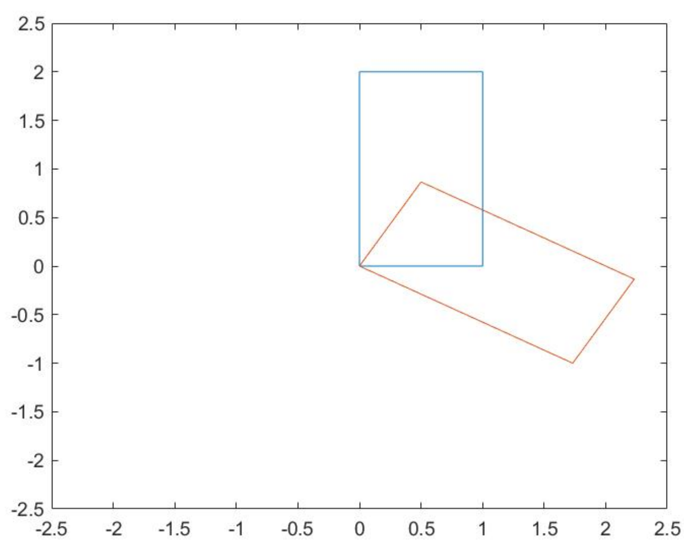


Figure 3.3: Reflection over 30 degree line.

Notice that, if we plug in $\theta = 90$, we get the matrix from part 1, which reflects over the x-axis (i.e., 90 degree line) and, if we plug in $\theta = 0$, we get the matrix from part 2, which reflects over the y-axis (i.e., the 0 degree line).

Solution 3.10

1. This shearing matrix pulls the points along horizontal lines and the strength of the pull is proportional to the y coordinate. In the figure below, the blue rectangle is sheared to become the orange rectangle:

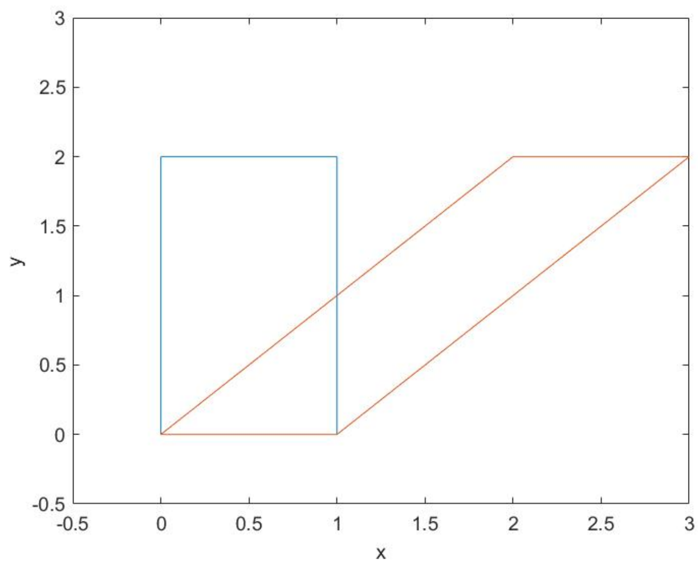


Figure 3.4: Shearing in x direction.

The area of the rectangle does not change. The inverse is

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

2. This shearing matrix pulls the points along vertical lines and the strength of the pull is proportional to the x coordinate. In the figure below, the blue rectangle is sheared to become the orange rectangle:

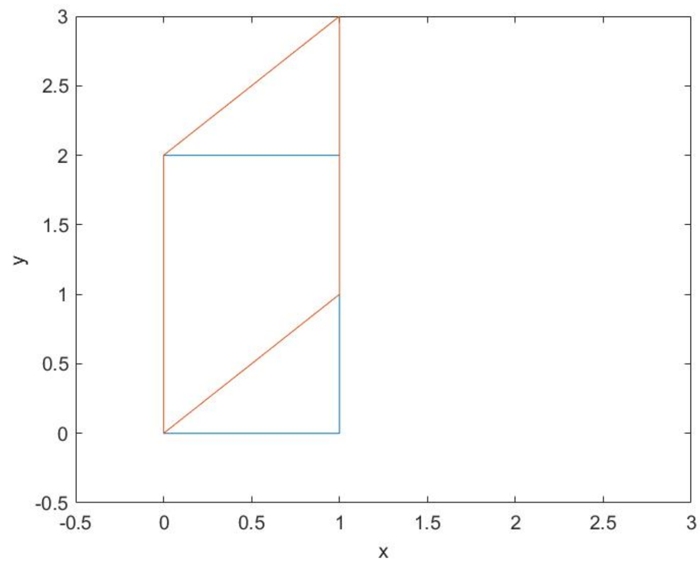


Figure 3.5: Shearing in y direction.

The area of the rectangle does not change. The inverse is

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

3. This shearing matrix pulls the points along horizontal lines and the strength of the pull is proportional to the y coordinate and the constant k (the bigger the k, the stronger the pull). In the figure below, with $k = 2$, the blue rectangle is sheared to become the orange rectangle:

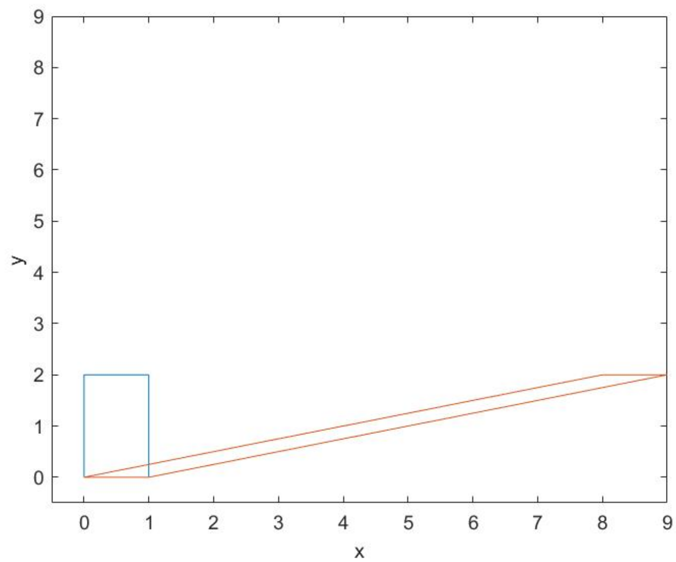


Figure 3.6: Shearing in x direction with $k = 2$.

The area of the rectangle does not change. The inverse is

$$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}.$$

4. This shearing matrix pulls the points along vertical lines and the strength of the pull is proportional to the x coordinate and the constant k (the bigger the k, the stronger the pull). In the figure below, with $k = 2$, the blue rectangle is sheared to become the orange rectangle:

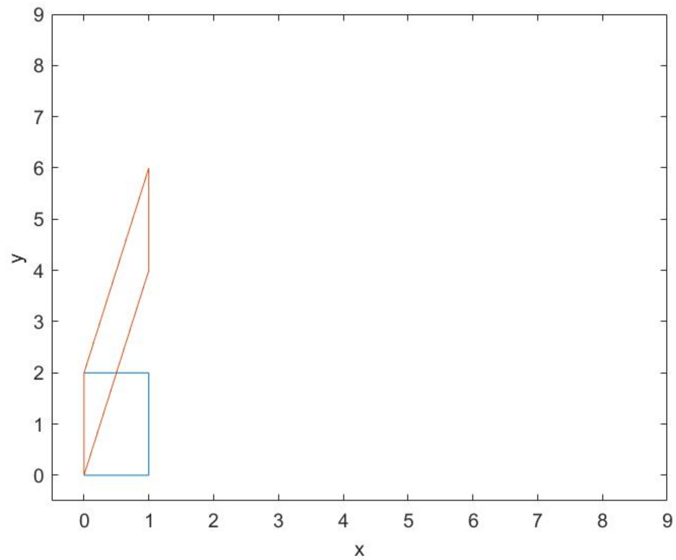


Figure 3.7: Shearing in y direction with $k = 2$.

The area of the rectangle does not change. The inverse is

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}.$$