Chapter 4

Night 2: Matrix Operations

Overview and Orientation

Learning Objectives

Concepts

- Compute the determinant of a 2×2 matrix
- Know the relationship between the determinant of a matrix and whether the matrix is invertible
- Find the inverse of a 2×2 matrix by hand
- Use computational tools to find the inverse of an $n \times n$ matrix
- Design a 2 or 3-dimensional matrix that will scale a vector by given amounts in the x, y or z direction
- Design a 3-dimensional matrix that will translate a 2-D vector by given amounts in x and y

MATLAB skills

- Represent a set of points in 2-D space (i.e., pairs of x, y values) as column vectors
- Transform a set of 2-D points (i.e., the outline of a shape) using a matrix to rotate and translate the original
- Multiply matrices and find their inverses
- Compute the determinant of a matrix

Suggested Approach

See Night 1 assignment for our general suggested approach to night assignments and a list of linear algebra resources.

4.1 Determinant of a Matrix

The determinant of a square matrix is a property of the matrix which indicates many important things, including whether a matrix is invertible or not. We will see more of this when we see matrix inverses shortly. The determinant of a matrix $\mathbf{G}$ is denoted a few different ways.

$$\det(\mathbf{G}) = |\mathbf{G}| \tag{4.1}$$

Consider a generic 2×2 matrix $\mathbf{G}$:

$$\mathbf{G} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The formula for the determinant of a 2×2 matrix is quite straightforward:

$$\det(\mathbf{G}) = ad - bc \tag{4.2}$$

For example, for the following 2×2 matrix,

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) &= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \\ &= (1)(4) - (2)(3) = -2 \end{aligned} \tag{4.3}$$

Exercise 4.1

Return to the transformation matrices in the day assignment and calculate the determinant for the following:

1. The generic 2×2 rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2. The matrix which reflects over the y axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. The matrix which shears in the horizontal direction

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise 4.2

1. What do the following matrices do? Think about it first, draw some sketches and then test your hypothesis in MATLAB. How much does the area of your basic rectangle change, if at all?

a)

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

2. Is it possible to "undo" the matrices above? Why or why not?

Exercise 4.3

1. What are the determinants of the two matrices from the previous exercise, Exercise 4.2?
2. Generalizing from Exercise 4.1 and Exercise 4.2, what's the relationship between the determinant of a matrix and the result of transforming a rectangle by that matrix?

Finding the determinant of an $n \times n$ matrix, where $n > 2$, is a bit more computationally intensive. If you want to learn how to do the procedure by hand, check out this Khan Academy video. For this course, we simply recommend you use the `det` function in MATLAB.

4.2 Matrix Inverses

Inverse of 2×2 *Matrices*

In class you worked with rotation matrices and transformations that were compositions of simpler rotations, and you learned how to invert them. When you multiply a vector by any matrix (not just ones that are associated with simple spatial transformations), you transform the original vector into a new vector. More generally (than rotations), you can *often* undo the linear transformation (just like you did with the rotation matrix). Undoing this linear transformation is a linear transformation itself! Therefore the act of undoing a linear transformation can be formulated with a matrix multiply.

Exercise 4.4

Consider the following matrices and vector. (Don't try to interpret these as intuitive geometrical operations; we're just using them to explore the determinant.) Work out the following problems in

MATLAB.

$$\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \tag{4.4}$$

$$\mathbf{Q} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix} \tag{4.5}$$

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{4.6}$$

1. Find $\mathbf{w} = \mathbf{Pu}$.
2. Find $\mathbf{Qw}$. How is this related to $\mathbf{u}$?
3. Find $\mathbf{QP}$. Does the answer look familiar?
4. Find $\mathbf{PQ}$.
5. Find the determinant of $\mathbf{P}$. In MATLAB, you can compute the determinant of any (not just 2×2) matrix using the `det` function.
6. Find the determinant of $\mathbf{Q}$.

A matrix $\mathbf{B}$ is said to be the inverse of the matrix $\mathbf{A}$ if, and only if, $\mathbf{BA} = \mathbf{I}$ and $\mathbf{AB} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. For 2×2 matrices, the inverse (if it exists) is given by the following

$$\mathbf{G} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4.7}$$

$$\mathbf{G}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{4.8}$$

The last equation should indicate to you that the inverse of the matrix $\mathbf{G}^{-1}$ is only defined if $ad - bc \neq 0$. Sweet mother of linear algebra, $ad - bc$ is our buddy the determinant. More generally, any square matrix can be inverted if and only if its determinant is non-zero.

Now let's practice calculating inverses, some of their properties, and how we may use them.

Exercise 4.5

All matrices $\mathbf{A}$ and $\mathbf{B}$ which have inverses have the following properties

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$
$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

1. Using the above properties, please compute the following by hand.

a) If

$$\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \tag{4.9}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \tag{4.10}$$

$$\tag{4.11}$$

find $(\mathbf{PB})^{-1}$. Recall that you already know the inverse of $\mathbf{P}$ from earlier.

b) For $\mathbf{P}$ as defined above, find

$$(\mathbf{P}^T)^{-1} \tag{4.12}$$

2. Use the inverse formula to calculate the inverses for the first three matrices in Exercise 4.1. Confirm your answers by multiplying the inverse with the original matrix.

   a) By hand, write an equation relating $\mathbf{n}$ and $\mathbf{d}$, using a matrix-vector product.

   b) By hand, calculate how many oranges and apples you have.

   c) Why do you think this type of problem is often called an inverse problem?

Note that solving matrix-vector equations like above can be done without explicitly computing the matrix inverse which is computationally expensive. (A nod to our future friend, left matrix divide or backslash divide.)

Inverse of $n \times n$ Matrices

For higher-dimensional matrices, e.g. $n \times n$ matrices for $n > 2$, the matrix inverse is defined in the same way. Suppose you have an $n \times n$ matrix $\mathbf{A}$ and an $n \times n$ matrix $\mathbf{B}$. Then $\mathbf{B}$ is the inverse of $\mathbf{A}$ if and only if $\mathbf{BA} = \mathbf{I}$ and $\mathbf{AB} = \mathbf{I}$. The following are some properties of inverses of matrices

- Only square matrices are invertible, i.e., only square matrices have inverses.
- A matrix has an inverse only if its determinant is non-zero.

There are a number of different procedures to compute the inverse of higher-dimensional matrices, but we will not be going into the details of their computation here. You can look them up if you are interested, or need to in the future. In MATLAB, you can compute the inverse of a matrix using the `inv` function.

Exercise 4.6

1. Consider the example with the fruits that you worked out earlier. Now, in addition to apples

and oranges, suppose you also had an unknown number of pears which each weigh 3 oz, and cost \$3. Additionally, suppose that the total weight of the fruits is 45 oz, and you paid a total of \$21 for the fruit.

a) If possible find the numbers of oranges, apples and pears. If not, please explain why.
b) Suppose that you additionally know that you have a total of 14 fruits. Can you formulate and solve a matrix-vector equation to find out the numbers of oranges, apples and pears you have?
c) What is the determinant of the matrix you have set up to solve this?

2. The fruit vendors bought the pricing algorithm from Uber. Oranges are still \$2, pears are now only \$1.50, and (due to an influx of teachers) apples are now surging at \$1.50 each. Their weights stay the same. You return to the market, and again purchase 14 fruits, which have the same total weight and total cost.

   a) Can you formulate and solve a matrix-vector equation to find out the numbers of oranges, apples and pears you have?
   b) What is the determinant of the matrix you have set up to solve this?
   c) Debrief at your table about what this means.

4.3 *Transformation Matrices, Continued*

Scaling

Returning to two dimensions. In the Night 1 assignment, you also learned about scaling matrices. Recall that the scaling matrix $\mathbf{S}$ scales the x-component by s_1 and the y-component by s_2

$$\mathbf{S} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}.$$

Let's assume for the moment that $s_1 = 2$ and $s_2 = 1/3$. Working with the rectangles defined in class whose corners have coordinates $(1, 2)$, $(1, -2)$, $(-1, 2)$, and $(-1, -2)$ complete the following activities:

Exercise 4.7

1. Predict what would happen if you operate on the rectangle with $\mathbf{S}$.
2. Write a MATLAB script to carry out this operation and check your prediction.
3. How does the area of the rectangle change?

4. What matrix should you use to *undo* this scaling? Show that the product of this matrix with the original scaling matrix is the *identity* matrix.
5. Define it in MATLAB and check. Again, this is the *inverse* matrix and we give it the symbol $\mathbf{S}^{-1}$.
6. In MATLAB, change the value of s_2 to 1 and find the product of the new $\mathbf{S}$ and your rectangle. How does the area of the rectangle change? Change the value of s_2 back to 1/3.
7. Predict what would happen if you operate on the original rectangle with $\mathbf{SR}$, where $\mathbf{R}$ is the rotation matrix. How about $\mathbf{RS}$? Implement both of these in MATLAB and check.
8. How would you *undo* each of these operations ($\mathbf{SR}$ and $\mathbf{RS}$)? How is the inverse of the product related to the individual inverses, i.e. what is the relationship between $(\mathbf{SR})^{-1}$ and $\mathbf{S}^{-1}$ and $\mathbf{R}^{-1}$? What about $(\mathbf{RS})^{-1}$?

Translation

It would be really useful if, in addition to scaling and rotating our objects, we could translate them. Let's start by thinking about vectors and then we will figure out how to represent translation as a matrix operation.

Consider an initial vector $\mathbf{v}$ and a translation vector $\mathbf{t}$. The new translated vector is simply $\mathbf{v} + \mathbf{t}$. For example, if you start with the initial vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and translate it using the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then the new vector is just $\begin{bmatrix} x+2 \\ y+3 \end{bmatrix}$. More generally, if the translation vector is $\begin{bmatrix} t_x \\ t_y \end{bmatrix}$ then the new vector will be $\begin{bmatrix} x+t_x \\ y+t_y \end{bmatrix}$.

Wouldn't it be handy if we could define translation as a matrix operation? Yes, indeed it would be, we hear you say. Here is the standard method: add another entry to the original vector, and set it equal to 1, i.e., $\mathbf{v} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. Now define the translation matrix as

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 4.8

1. Show that $\mathbf{Tv}$ accomplishes the process of translation (if you ignore the third entry in the new vector). What is the final vector?
2. Predict what would happen if you operate on our old friend the rectangle with the translation

matrix defined by $t_x = 2$ and $t_y = 3$.

3. Write a MATLAB script to carry out this operation and check your prediction. How has the area of your rectangle changed?

4. What matrix should you use to *undo* this translation? Show on paper that the product of this matrix with the original translation matrix is the *identity* matrix. Define it in MATLAB and check. Again, this is the *inverse* matrix and we give it the symbol $\mathbf{T}^{-1}$.

5. Choose a rotation matrix $\mathbf{R}$. Predict what would happen if you operate on the original rectangle with $\mathbf{TR}$. How about $\mathbf{RT}$? Implement both of these in MATLAB and check. How would you undo each of these operations? (You will first have to adjust your definition of $\mathbf{R}$ so that it is the correct size.)

6. Predict what would happen if you operate on the original rectangle with $\mathbf{STR}$. How about $\mathbf{TRS}$? How would you *undo* each of these operations? (You will first have to adjust your definition of $\mathbf{S}$ so that it is the correct size.)

7. How would you generalize translation to 3D?

Putting it all together: Dancing Animals

In this activity you will animate a circus act. (No real or imaginary animals will be injured in this performance.) Here is what we would like you to do:

Exercise 4.9

1. Decide on an animal.

2. Decide on a circus act that consists of a set of translations, rotations (think back to Day 2), shearings, and/or scalings in some order. Storyboard this idea and imagine the resulting animation.

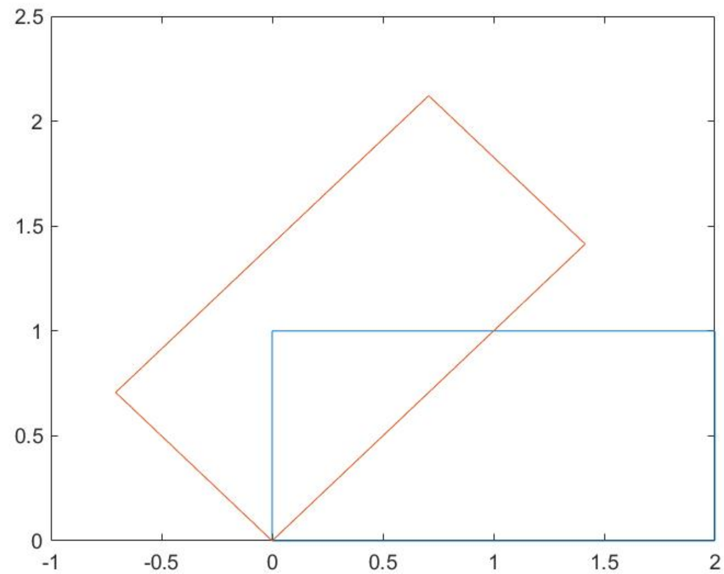
3. Propose a set of points that defines the outline and relevant features of your animal. You may find `ginput` useful. Define the points in MATLAB and plot your animal.

4. Create a script that makes your animal dance (in 2-D, unless you really want to go 3-D). You may want to make use of the `pause` and `drawnow` commands.

5. Now use your sequence of operations and animate your animal! In class you will have the opportunity to show off your dancing animal!

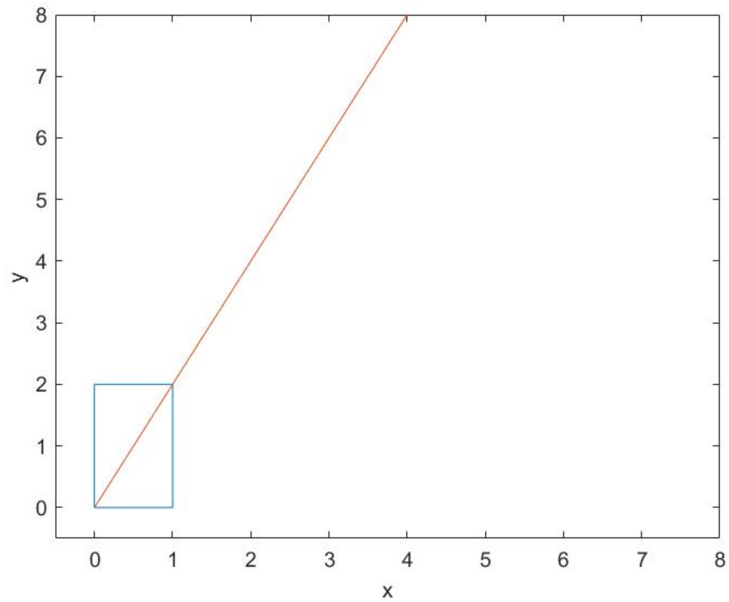
4.4 Conceptual Quiz

1. The orange shape is the result of applying a matrix $\mathbf{M}$ to the blue rectangle.



   What is the determinant of $\mathbf{M}$?

2. The orange shape is the result of applying a matrix $\mathbf{M}$ to the blue rectangle.



What is the determinant of $\mathbf{M}$?

3. The determinant is multiplicative, i.e, $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$. Let $\mathbf{M}$ be a matrix such that $\det(\mathbf{M}) = \frac{1}{3}$. What's $\det(\mathbf{M}^{-1})$? (Hint: $\det(\mathbf{I}) = 1$.)

4. Let R be a rectangle with area 1. Apply the scaling matrix $\mathbf{S} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$. What is the area of $\mathbf{S}R$?

   A. $\frac{s_1 s_2}{2}$
   B. 1
   C. $s_1 s_2$
   D. $s_1 + s_2$

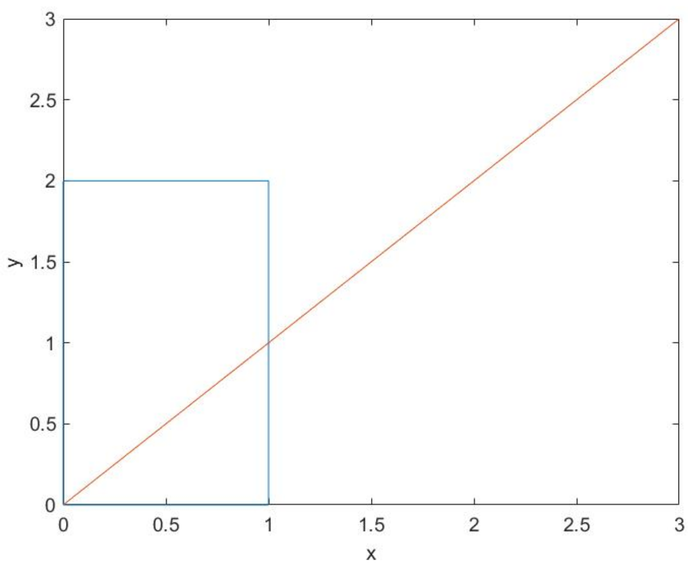
5. True or false: Any shearing matrix $\mathbf{S}$ and any rotation matrix $\mathbf{R}$ commute, i.e., $\mathbf{RS} = \mathbf{SR}$.

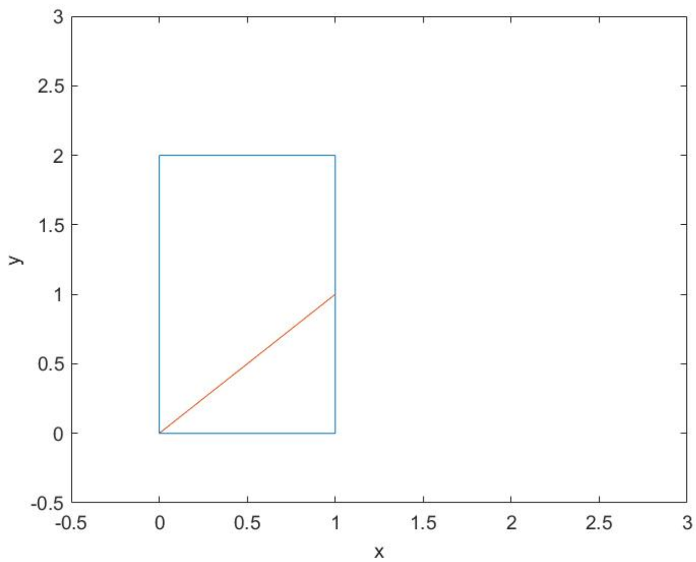
Solution 4.1

1. The determinant is 1. (Recall that $\cos^2\theta + \sin^2\theta = 1$.)
2. The determinant is -1.
3. The determinant is 1.

Solution 4.2

1. Each of the figures below shows the basic blue rectangle and the orange rectangle, which is the result of applying the transformation.

a) 



b)

2. It is not possible to undo these matrix transformations. Since everything is squished onto the same line, we would not be able to distinguish the original vectors.

Notice that, in the above matrices, the first row is a constant multiple of the second row. In other words, the matrix looks like $\begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$ for some constant c. If we apply a matrix of this form to a point in 2D space reprsented by the vector $\begin{bmatrix} x \\ y \end{bmatrix}$, then the result will be $\begin{bmatrix} z \\ cz \end{bmatrix}$, where $z = ax + by$. In other words, the resulting point will always fall on the line $y = cx$.

Solution 4.4

1.
$$\mathbf{w} = \mathbf{P}\mathbf{u} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

2.
$$\mathbf{Q}\mathbf{w} = \mathbf{Q}\mathbf{P}\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

3.
$$\mathbf{Q}\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   which is the identity matrix

4. The determinate of $\mathbf{P}$ is 2.

5. The determinate of $\mathbf{Q}$ is $\frac{1}{2}$.

Solution 4.5

1. a)

$$(\mathbf{PB})^{-1} = \begin{bmatrix} -17/2 & 7/2 \\ 5 & -2 \end{bmatrix}$$

   b)

$$(\mathbf{P}^T)^{-1} = \begin{bmatrix} 3/2 & -2 \\ -1/2 & 1 \end{bmatrix}$$

2.

$$\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Solution 4.6

1. a) It's not possible to find the numbers of oranges, apples, and pears. We have the equation

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 3 \end{bmatrix} \begin{bmatrix} n_0 \\ n_a \\ n_p \end{bmatrix} = \begin{bmatrix} 21 \\ 45 \end{bmatrix},$$

   but we cannot take the inverse of a 2×3 (non-square) matrix.

   b) Now we have the equation

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} n_0 \\ n_a \\ n_p \end{bmatrix} = \begin{bmatrix} 21 \\ 45 \\ 14 \end{bmatrix}.$$

   So by taking the inverse of the 3×3 matrix we find that $n_0 = 3, n_a = 9$ and $n_p = 2$.

   c) The determinant of the matrix is 2.

2. a) The equation becomes

$$\begin{bmatrix} 2 & \frac{3}{2} & \frac{3}{2} \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} n_0 \\ n_a \\ n_p \end{bmatrix} = \begin{bmatrix} 21 \\ 45 \\ 14 \end{bmatrix}.$$

   But the matrix is not invertible, so we cannot solve for the number of fruit.

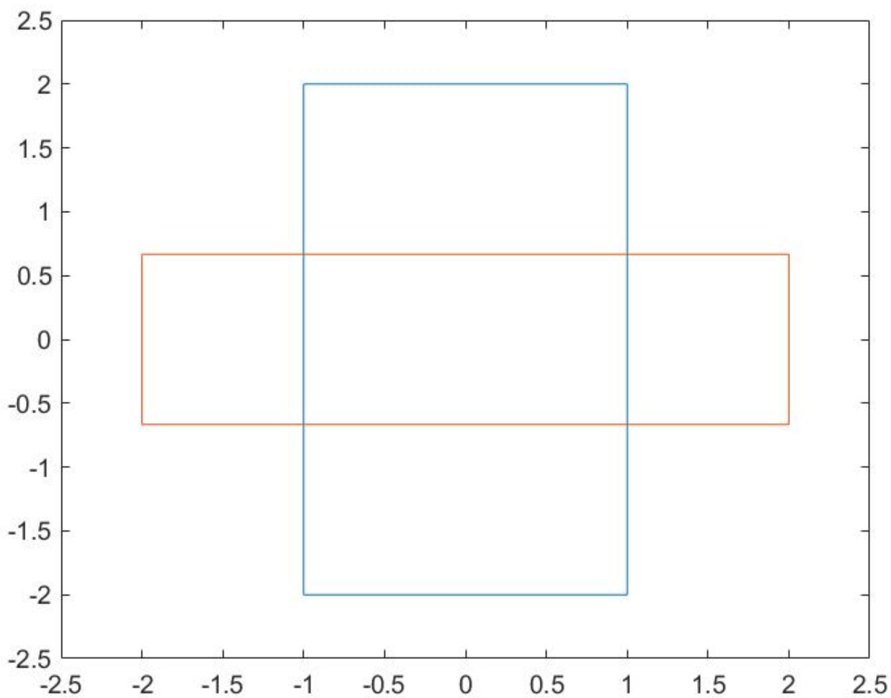
   b) The determinant of the matrix is 0.

   c)

Solution 4.7

1. The length of the rectangle would double in the x direction and be reduced to 1/3 the length in the y direction.

2. First we define the corners of the rectangle as the columns in a matrix
   » `points=[1 1 -1 -1; 2 -2 -2 2]`
   and we define the scaling matrix
   » `S=[2 0; 0 1/3]`. Then we simply multiply them
   » `scaledpoint=S*points`.
   Plotting them, here is the original rectangle in blue and the scaled rectangle in orange



3. The area is reduced from 8 units^2 to 5.33 units^2, or 2/3 of the original area.

4. To undo the process we use the inverse of the $\mathbf{S}$ matrix, or $\mathbf{S}^{-1}$ would be used.

$$\mathbf{S}^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 3 \end{bmatrix}.$$

   You should check that $\mathbf{S}^{-1}\mathbf{S} = \mathbf{S}\mathbf{S}^{-1} = \mathbf{I}$.

5. We define the inverse matrix » `Sinv=[0.5 0; 0 3]` and check that » `S*Sinv` and `Sinv*S` both produce the identity matrix.

6. The area of the rectangle doubles.

7. When the original rectangle is operated on with

$$\mathbf{SR}$$

, the resulting image will be a horizontally stretched parallelogram. When the original rectangle is operated on with $\mathbf{RS}$, the resulting image will be the scaled rectangle from the previous exercise only rotated 60 degrees counter-clockwise.

8. $(\mathbf{SR})^{-1} = \mathbf{R}^{-1}\mathbf{S}^{-1}$ or $(\mathbf{RS})^{-1} = \mathbf{S}^{-1}\mathbf{R}^{-1}$

Solution 4.8

1. $\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$

2. The rectangle would be moved 2 to the right and 3 up.

3. The area of the rectangle does not change.

4.
$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

5. If the original rectangle is operated on by $\mathbf{TR}$, the rectangle would first be rotated with respect to the origin and than translated. If the original rectangle is operated on by $\mathbf{TR}$, the rectangle would first be translated and then rotated. As rotation happens with respect to the origin, the 2 operations will not result in the same rectangle.

   To undo the operation $\mathbf{TR}$, the resulting figure should be operated on by $\mathbf{R}^{-1}\mathbf{T}^{-1}$. To undo the operation $\mathbf{RT}$, the resulting figure should be operated on by $\mathbf{T}^{-1}\mathbf{R}^{-1}$.

6. If the original rectangle is operated on with $\mathbf{STR}$, the resulting image will be of the rectangle rotated 60 degrees around the origin, translated 2 to the right and 3 up and then scaled by $\mathbf{S}$. If the original rectangle is operated on with $\mathbf{TRS}$, the resulting image will be the scaled rectangle rotated 60 degrees around the origin and then translated 2 to the right and 3 up.

   To undo $\mathbf{STR}$, the resulting figure should be operated on by $\mathbf{R}^{-1}\mathbf{T}^{-1}\mathbf{S}^{-1}$. To undo $\mathbf{TRS}$, the resulting figure should be operated on by $\mathbf{S}^{-1}\mathbf{R}^{-1}\mathbf{T}^{-1}$.

7.
$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$